

# Solving quadratic equations by factorisation

## A LEVEL LINKS

Scheme of work: 1b. Quadratic functions – factorising, solving, graphs and the discriminants

## Key points

- A quadratic equation is an equation in the form  $ax^2 + bx + c = 0$  where  $a \neq 0$ .
- To factorise a quadratic equation find two numbers whose sum is  $b$  and whose products is  $ac$ .
- When the product of two numbers is 0, then at least one of the numbers must be 0.
- If a quadratic can be solved it will have two solutions (these may be equal).

## Examples

**Example 1** Solve  $5x^2 = 15x$

$5x^2 = 15x$ $5x^2 - 15x = 0$ $5x(x - 3) = 0$ <p>So <math>5x = 0</math> or <math>(x - 3) = 0</math></p> <p>Therefore <math>x = 0</math> or <math>x = 3</math></p>	<ol style="list-style-type: none"> <li>1 Rearrange the equation so that all of the terms are on one side of the equation and it is equal to zero. Do not divide both sides by <math>x</math> as this would lose the solution <math>x = 0</math>.</li> <li>2 Factorise the quadratic equation. <math>5x</math> is a common factor.</li> <li>3 When two values multiply to make zero, at least one of the values must be zero.</li> <li>4 Solve these two equations.</li> </ol>
-------------------------------------------------------------------------------------------------------------------------------------------------------------------	-------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------

**Example 2** Solve  $x^2 + 7x + 12 = 0$

$x^2 + 7x + 12 = 0$ $b = 7, ac = 12$ $x^2 + 4x + 3x + 12 = 0$ $x(x + 4) + 3(x + 4) = 0$ $(x + 4)(x + 3) = 0$ <p>So <math>(x + 4) = 0</math> or <math>(x + 3) = 0</math></p> <p>Therefore <math>x = -4</math> or <math>x = -3</math></p>	<ol style="list-style-type: none"> <li>1 Factorise the quadratic equation. Work out the two factors of <math>ac = 12</math> which add to give you <math>b = 7</math>. (4 and 3)</li> <li>2 Rewrite the <math>b</math> term (<math>7x</math>) using these two factors.</li> <li>3 Factorise the first two terms and the last two terms.</li> <li>4 <math>(x + 4)</math> is a factor of both terms.</li> <li>5 When two values multiply to make zero, at least one of the values must be zero.</li> <li>6 Solve these two equations.</li> </ol>
-----------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------	-----------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------

**Example 3** Solve  $9x^2 - 16 = 0$

$9x^2 - 16 = 0$ $(3x + 4)(3x - 4) = 0$ <p>So <math>(3x + 4) = 0</math> or <math>(3x - 4) = 0</math></p> $x = -\frac{4}{3} \text{ or } x = \frac{4}{3}$	<ol style="list-style-type: none"> <li><b>1</b> Factorise the quadratic equation. This is the difference of two squares as the two terms are <math>(3x)^2</math> and <math>(4)^2</math>.</li> <li><b>2</b> When two values multiply to make zero, at least one of the values must be zero.</li> <li><b>3</b> Solve these two equations.</li> </ol>
--------------------------------------------------------------------------------------------------------------------------------------------------------	--------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------

**Example 4** Solve  $2x^2 - 5x - 12 = 0$

$b = -5, ac = -24$ <p>So <math>2x^2 - 8x + 3x - 12 = 0</math></p> $2x(x - 4) + 3(x - 4) = 0$ $(x - 4)(2x + 3) = 0$ <p>So <math>(x - 4) = 0</math> or <math>(2x + 3) = 0</math></p> $x = 4 \text{ or } x = -\frac{3}{2}$	<ol style="list-style-type: none"> <li><b>1</b> Factorise the quadratic equation. Work out the two factors of <math>ac = -24</math> which add to give you <math>b = -5</math>. (-8 and 3)</li> <li><b>2</b> Rewrite the <math>b</math> term (<math>-5x</math>) using these two factors.</li> <li><b>3</b> Factorise the first two terms and the last two terms.</li> <li><b>4</b> <math>(x - 4)</math> is a factor of both terms.</li> <li><b>5</b> When two values multiply to make zero, at least one of the values must be zero.</li> <li><b>6</b> Solve these two equations.</li> </ol>
-------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------	-----------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------

## Practice

**1** Solve

- |                                                                                                                                                                                                                                                                                       |                                                                                                                                                                                                                                                                                    |
|---------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------|------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------|
| <p><b>a</b> <math>6x^2 + 4x = 0</math></p> <p><b>c</b> <math>x^2 + 7x + 10 = 0</math></p> <p><b>e</b> <math>x^2 - 3x - 4 = 0</math></p> <p><b>g</b> <math>x^2 - 10x + 24 = 0</math></p> <p><b>i</b> <math>x^2 + 3x - 28 = 0</math></p> <p><b>k</b> <math>2x^2 - 7x - 4 = 0</math></p> | <p><b>b</b> <math>28x^2 - 21x = 0</math></p> <p><b>d</b> <math>x^2 - 5x + 6 = 0</math></p> <p><b>f</b> <math>x^2 + 3x - 10 = 0</math></p> <p><b>h</b> <math>x^2 - 36 = 0</math></p> <p><b>j</b> <math>x^2 - 6x + 9 = 0</math></p> <p><b>l</b> <math>3x^2 - 13x - 10 = 0</math></p> |
|---------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------|------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------|

**2** Solve

- |                                                                                                                                                                                         |                                                                                                                                                                                      |
|-----------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------|--------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------|
| <p><b>a</b> <math>x^2 - 3x = 10</math></p> <p><b>c</b> <math>x^2 + 5x = 24</math></p> <p><b>e</b> <math>x(x + 2) = 2x + 25</math></p> <p><b>g</b> <math>x(3x + 1) = x^2 + 15</math></p> | <p><b>b</b> <math>x^2 - 3 = 2x</math></p> <p><b>d</b> <math>x^2 - 42 = x</math></p> <p><b>f</b> <math>x^2 - 30 = 3x - 2</math></p> <p><b>h</b> <math>3x(x - 1) = 2(x + 1)</math></p> |
|-----------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------|--------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------|

**Hint**

Get all terms onto one side of the equation.

# Solving quadratic equations by completing the square

## A LEVEL LINKS

Scheme of work: 1b. Quadratic functions – factorising, solving, graphs and the discriminants

## Key points

- Completing the square lets you write a quadratic equation in the form  $p(x + q)^2 + r = 0$ .

## Examples

**Example 5** Solve  $x^2 + 6x + 4 = 0$ . Give your solutions in surd form.

$x^2 + 6x + 4 = 0$ $(x + 3)^2 - 9 + 4 = 0$ $(x + 3)^2 - 5 = 0$ $(x + 3)^2 = 5$ $x + 3 = \pm\sqrt{5}$ $x = \pm\sqrt{5} - 3$ <p>So <math>x = -\sqrt{5} - 3</math> or <math>x = \sqrt{5} - 3</math></p>	<ol style="list-style-type: none"> <li>Write <math>x^2 + bx + c = 0</math> in the form <math>\left(x + \frac{b}{2}\right)^2 - \left(\frac{b}{2}\right)^2 + c = 0</math></li> <li>Simplify.</li> <li>Rearrange the equation to work out <math>x</math>. First, add 5 to both sides.</li> <li>Square root both sides. Remember that the square root of a value gives two answers.</li> <li>Subtract 3 from both sides to solve the equation.</li> <li>Write down both solutions.</li> </ol>
------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------	-------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------

**Example 6** Solve  $2x^2 - 7x + 4 = 0$ . Give your solutions in surd form.

$2x^2 - 7x + 4 = 0$ $2\left(x^2 - \frac{7}{2}x\right) + 4 = 0$ $2\left[\left(x - \frac{7}{4}\right)^2 - \left(\frac{7}{4}\right)^2\right] + 4 = 0$ $2\left(x - \frac{7}{4}\right)^2 - \frac{49}{8} + 4 = 0$ $2\left(x - \frac{7}{4}\right)^2 - \frac{17}{8} = 0$	<ol style="list-style-type: none"> <li>Before completing the square write <math>ax^2 + bx + c</math> in the form <math>a\left(x^2 + \frac{b}{a}x\right) + c</math></li> <li>Now complete the square by writing <math>x^2 - \frac{7}{2}x</math> in the form <math>\left(x + \frac{b}{2a}\right)^2 - \left(\frac{b}{2a}\right)^2</math></li> <li>Expand the square brackets.</li> <li>Simplify.</li> </ol> <p style="text-align: right;"><i>(continued on next page)</i></p>
------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------	----------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------

$2\left(x - \frac{7}{4}\right)^2 = \frac{17}{8}$ $\left(x - \frac{7}{4}\right)^2 = \frac{17}{16}$ $x - \frac{7}{4} = \pm \frac{\sqrt{17}}{4}$ $x = \pm \frac{\sqrt{17}}{4} + \frac{7}{4}$ <p>So <math>x = \frac{7}{4} - \frac{\sqrt{17}}{4}</math> or <math>x = \frac{7}{4} + \frac{\sqrt{17}}{4}</math></p>	<p><b>5</b> Rearrange the equation to work out <math>x</math>. First, add <math>\frac{17}{8}</math> to both sides.</p> <p><b>6</b> Divide both sides by 2.</p> <p><b>7</b> Square root both sides. Remember that the square root of a value gives two answers.</p> <p><b>8</b> Add <math>\frac{7}{4}</math> to both sides.</p> <p><b>9</b> Write down both the solutions.</p>
--------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------	-------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------

## Practice

**3** Solve by completing the square.

**a**  $x^2 - 4x - 3 = 0$

**c**  $x^2 + 8x - 5 = 0$

**e**  $2x^2 + 8x - 5 = 0$

**b**  $x^2 - 10x + 4 = 0$

**d**  $x^2 - 2x - 6 = 0$

**f**  $5x^2 + 3x - 4 = 0$

**4** Solve by completing the square.

**a**  $(x - 4)(x + 2) = 5$

**b**  $2x^2 + 6x - 7 = 0$

**c**  $x^2 - 5x + 3 = 0$

### Hint

Get all terms onto one side of the equation.

# Solving quadratic equations by using the formula

## A LEVEL LINKS

Scheme of work: 1b. Quadratic functions – factorising, solving, graphs and the discriminants

## Key points

- Any quadratic equation of the form  $ax^2 + bx + c = 0$  can be solved using the formula

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

- If  $b^2 - 4ac$  is negative then the quadratic equation does not have any real solutions.
- It is useful to write down the formula before substituting the values for  $a$ ,  $b$  and  $c$ .

## Examples

**Example 7** Solve  $x^2 + 6x + 4 = 0$ . Give your solutions in surd form.

$$a = 1, b = 6, c = 4$$

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

$$x = \frac{-6 \pm \sqrt{6^2 - 4(1)(4)}}{2(1)}$$

$$x = \frac{-6 \pm \sqrt{20}}{2}$$

$$x = \frac{-6 \pm 2\sqrt{5}}{2}$$

$$x = -3 \pm \sqrt{5}$$

$$\text{So } x = -3 - \sqrt{5} \text{ or } x = \sqrt{5} - 3$$

- Identify  $a$ ,  $b$  and  $c$  and write down the formula.

Remember that  $-b \pm \sqrt{b^2 - 4ac}$  is all over  $2a$ , not just part of it.

- Substitute  $a = 1$ ,  $b = 6$ ,  $c = 4$  into the formula.

- Simplify. The denominator is 2, but this is only because  $a = 1$ . The denominator will not always be 2.

- Simplify  $\sqrt{20}$ .

$$\sqrt{20} = \sqrt{4 \times 5} = \sqrt{4} \times \sqrt{5} = 2\sqrt{5}$$

- Simplify by dividing numerator and denominator by 2.

- Write down both the solutions.

**Example 8** Solve  $3x^2 - 7x - 2 = 0$ . Give your solutions in surd form.

$a = 3, b = -7, c = -2$ $x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$ $x = \frac{-(-7) \pm \sqrt{(-7)^2 - 4(3)(-2)}}{2(3)}$ $x = \frac{7 \pm \sqrt{73}}{6}$ <p>So <math>x = \frac{7 - \sqrt{73}}{6}</math> or <math>x = \frac{7 + \sqrt{73}}{6}</math></p>	<ol style="list-style-type: none"> <li><b>1</b> Identify <math>a</math>, <math>b</math> and <math>c</math>, making sure you get the signs right and write down the formula. Remember that <math>-b \pm \sqrt{b^2 - 4ac}</math> is all over <math>2a</math>, not just part of it.</li> <li><b>2</b> Substitute <math>a = 3</math>, <math>b = -7</math>, <math>c = -2</math> into the formula.</li> <li><b>3</b> Simplify. The denominator is 6 when <math>a = 3</math>. A common mistake is to always write a denominator of 2.</li> <li><b>4</b> Write down both the solutions.</li> </ol>
-------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------	------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------

## Practice

**5** Solve, giving your solutions in surd form.

**a**  $3x^2 + 6x + 2 = 0$

**b**  $2x^2 - 4x - 7 = 0$

**6** Solve the equation  $x^2 - 7x + 2 = 0$

Give your solutions in the form  $\frac{a \pm \sqrt{b}}{c}$ , where  $a$ ,  $b$  and  $c$  are integers.

**7** Solve  $10x^2 + 3x + 3 = 5$

Give your solution in surd form.

**Hint**

Get all terms onto one side of the equation.

## Extend

**8** Choose an appropriate method to solve each quadratic equation, giving your answer in surd form when necessary.

**a**  $4x(x - 1) = 3x - 2$

**b**  $10 = (x + 1)^2$

**c**  $x(3x - 1) = 10$

## Answers

- 1 a**  $x = 0$  or  $x = -\frac{2}{3}$       **b**  $x = 0$  or  $x = \frac{3}{4}$   
**c**  $x = -5$  or  $x = -2$       **d**  $x = 2$  or  $x = 3$   
**e**  $x = -1$  or  $x = 4$       **f**  $x = -5$  or  $x = 2$   
**g**  $x = 4$  or  $x = 6$       **h**  $x = -6$  or  $x = 6$   
**i**  $x = -7$  or  $x = 4$       **j**  $x = 3$   
**k**  $x = -\frac{1}{2}$  or  $x = 4$       **l**  $x = -\frac{2}{3}$  or  $x = 5$
- 2 a**  $x = -2$  or  $x = 5$       **b**  $x = -1$  or  $x = 3$   
**c**  $x = -8$  or  $x = 3$       **d**  $x = -6$  or  $x = 7$   
**e**  $x = -5$  or  $x = 5$       **f**  $x = -4$  or  $x = 7$   
**g**  $x = -3$  or  $x = 2\frac{1}{2}$       **h**  $x = -\frac{1}{3}$  or  $x = 2$
- 3 a**  $x = 2 + \sqrt{7}$  or  $x = 2 - \sqrt{7}$       **b**  $x = 5 + \sqrt{21}$  or  $x = 5 - \sqrt{21}$   
**c**  $x = -4 + \sqrt{21}$  or  $x = -4 - \sqrt{21}$       **d**  $x = 1 + \sqrt{7}$  or  $x = 1 - \sqrt{7}$   
**e**  $x = -2 + \sqrt{6.5}$  or  $x = -2 - \sqrt{6.5}$       **f**  $x = \frac{-3 + \sqrt{89}}{10}$  or  $x = \frac{-3 - \sqrt{89}}{10}$
- 4 a**  $x = 1 + \sqrt{14}$  or  $x = 1 - \sqrt{14}$       **b**  $x = \frac{-3 + \sqrt{23}}{2}$  or  $x = \frac{-3 - \sqrt{23}}{2}$   
**c**  $x = \frac{5 + \sqrt{13}}{2}$  or  $x = \frac{5 - \sqrt{13}}{2}$
- 5 a**  $x = -1 + \frac{\sqrt{3}}{3}$  or  $x = -1 - \frac{\sqrt{3}}{3}$       **b**  $x = 1 + \frac{3\sqrt{2}}{2}$  or  $x = 1 - \frac{3\sqrt{2}}{2}$
- 6**  $x = \frac{7 + \sqrt{41}}{2}$  or  $x = \frac{7 - \sqrt{41}}{2}$
- 7**  $x = \frac{-3 + \sqrt{89}}{20}$  or  $x = \frac{-3 - \sqrt{89}}{20}$
- 8 a**  $x = \frac{7 + \sqrt{17}}{8}$  or  $x = \frac{7 - \sqrt{17}}{8}$   
**b**  $x = -1 + \sqrt{10}$  or  $x = -1 - \sqrt{10}$   
**c**  $x = -1\frac{2}{3}$  or  $x = 2$